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Exercise 1.5.13

a. $y_1^n = y_1 \cdot y_i \cdots \cdot y_1 = x$

$y_2^n = y_2 \cdot y_2 \cdot \ldots \cdot y_2 = x$

$y_1^n + (-y_2^n) = 0$ (Inv +)

$y_1(y_1^{n-1} + (-y_2^n)\cdot y^{-1}) = 0$ (D), (Inv ·)

$z \cdot 0 = 0$ (proposition)

$z + 0 = z(1+0)$ (Id+) (Id·)

$z + 0 = z\cdot 1 + z \cdot 0$

$z + 0 = z + z \cdot 0$ (Id·)

so $0 = z \cdot 0$ (cancellation)

$x + z = x + y \Rightarrow z = y$

$z = z + x + (-x)$ (Id+) (Inv+)

$z = (z + x) + (-x)$

$z = (x + y) + (-x)$ (assumption)

$z = y$ (Inv+)

$y_1^{n-1} + (-y_2^n)\cdot y_1^{-1} = 0$

let $(-y_2^n)\cdot y_1^{-1} = \alpha \cdot y_1^{-1}$

$\alpha = y_1 \cdot \beta$ (uniqness of Inv ·)

$\beta = y_1^{n-1}$ (uniqness of Inv+)

$\therefore -y_2^n = y_1^{n-1} \cdot y_1$

$= y_1^n$

(2) b)

Lemma:

prove if $x < y \Rightarrow xz < zy$ for $z > 0$

$(y - x) > 0$ (proposition)

$z \cdot (y - x) > 0$ ($OF\cdot$)

$zy - zx > 0$ (D)

$zy - zx + zx > zx + 0$ ($OF+$)

$zy > zx$ (Inv^+) (Id^+)

Base case:

$x(x+1)^{-1} \leq x$

$x^{-1} \cdot x \cdot (x+1)^{-1} \leq x \cdot x^{-1}$ (above)

$(x+1)^{-1} \leq 1$ ($Inv^\cdot$)

$(x+1)^{-1} \cdot (x+1) \leq 1 \cdot (x+1)$ (above)

$1 \leq x + 1$ ($Inv^\cdot$)

$1 + (-1) \leq x + 1 (-1)$ ($OF+$)

$0 \leq x$ (~~OF~~ Inv^+)

which is fine because we have that $x > 0$ in the prosposition

Inductive step

$$(x(x+1)^{-1})^{n+1} < (x(x+1))^{n}$$

if we multiply both sides by $((x(x+1))^{n})^{-1}$ n times we get

$$(x(x+1)^{-1})) < 1$$

$$x^{-1} \cdot x(x+1)^{-1} < x^{-1} \cdot 1 \quad (\text{above})$$

$$(x+1)^{-1} < x^{-1} \quad (\text{Inv}\cdot)(\text{Id}\cdot)$$

$$(x+1)^{-1} \cdot (x+1) < x^{-1} \cdot (x+1) \quad (\text{above})$$

$$1 < x^{-1} \cdot x + x^{-1} \quad (D)(\text{Inv}\cdot)$$

$$1 < 1 + x^{-1} \quad (\text{Inv}\cdot)$$

$$1 + (-1) < 1 + x^{-1} + (-1) \quad (\text{OF}+)$$

$$0 < x^{-1} \quad (\text{Inv}+)$$

if $x > 0 \Rightarrow x^{-1} > 0$ (proposition)

thus if $x > 0$, $x(x+1)^{-1}$ is always an element a A

③

c. u is an upper baund of A iff

$a \in A \,(a \le u)$ (definition)

$u = x + 1 > x$

$x + 1 + (-x) > x + (-x)$ (OF+)

$1 > 0$ (Inv+)

$1 \cdot 1 > 0$ (OF$\cdot$)

$1 \cdot 1 \cdot 1^{-1} > 0 \cdot (1^{-1})$ (above)

$1 > 0$ ($x \cdot 0 = 0$, proposition)

if $x + 1 > x$ (which it is as shown),

and $z < x$ $\Rightarrow z < x + 1$ (OP-t)

if $t^n \leq x \Rightarrow t^! \leq x$

by transitivity $t < x + 1$

then $x + 1$ is an upper bound

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Lemmas:

d.

Show that $x \geq 1$ if

$x \in A$, and $A = \{z \in F : zy \geq y\}$

$xy \geq y$ (assumption)

$xy \cdot y^{-1} \geq y \cdot y^{-1}$ (proposition)

$x \geq 1$ $\therefore$ elements of A are bigger than or equal to 1,

so $N \subset A$ and when $x = 1$

then $zy = y$ (uniqueness of $Id^{\cdot}$)

Show that $x > 0$ if $x \in A$

and $A = \{z \in F ; y + x > y\}$

$y + x > y$ (assumption)

$y + x + (-y) > y + (-y)$ (OF+)

$x > 0$ (Inv+)

$\therefore$ all $x > 0$

for $a = 0$:

$$b^n - a^n = (b-a)(b^{n-1})$$

we know that

$$b^n - a^n \geq (b-a)\,n\,(b^{n-1}) \quad \text{(above)}$$

and that

$$b^n - a^n > (b-a)\,n\,(b^{n-1}) \quad \text{if } n \neq 1 \quad \text{(above)}$$

we know that

$$b^{n-1} + \alpha > b^{n-1} \quad \text{(above)} \quad \text{if } \alpha > 0$$

let $\alpha = b^{n-2}a + \dots + ba^{n-2} + a^{n-1}$

$\alpha > 0$ (uniqness of Inv+) (OF $\cdot$)

since $a > 0 \wedge b > 0$

So then

$$(b-a)\,n\,(b^{n-1}) < (b-a)\,n\,(b^{n-1} + \alpha)$$

$$\therefore\; b^n - a^n < (b-a)\,n\,(b^{n-1} + \alpha) \quad \text{(PO-t)}$$

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e) $y^n - t^n < (y-t)\,n\,y^{n-1}$

$$t = y - K$$
$$= y - \frac{y^n - x}{ny^{n-1}}$$
$$= \frac{yn - y^n - x}{ny^{n-1}} \qquad (D)$$
$$= y - \frac{y}{n} + \frac{x}{ny^{n-1}}$$

So $y^n - t^n < \left(y - y + \frac{y}{n} - \frac{x}{ny^{n-1}}\right) n y^{n-1}$

$\therefore$ $y^n - t^n < y^n - x$

$-t^n < -x$ (OF+)

$x < t^n$ (OF+)

This holds because $t < y$:

$t = y - K = y - \left(\frac{y^n - x}{ny^{n-1}}\right)$

$\left(\frac{y^n - x}{ny^{n-1}}\right)$ is always positive because

$y^n - x > 0$ (assumption)

f. Assuming $y^n > x$ leads to a contradiction not because it implies t is an upper bound

$u \in F$ is an upper bound of A iff

$$\forall t \in A (t \leq u)$$

$t \leq t$ is an ordering axiom

The contradiction is that $t^n > x$, even though we initially constraind t^n to be lower than or equal to x, and found A to be non-empty.

⑥

g. What we are doing is setting h to all of the values $\frac{x-y^n}{n(y+1)^{n-1}}$ that are between 0 and 1.

This sets an upper band of the set of h elements to be 1.

$s = y + h$ is always positive, so we can use the inequality from question (d)

h. $s^n + y^n < (s-y)n\,s^{n-1}$

$s^n + y^n < (h+y-y)n\,s^{n-1}$

$s^n + y^n < hn\,s^{n-1}$

So $hn\,s^{n-1} < hn(y+1)^{n-1}$ because $s = y + h$ is bounded by $s = y + 1$, since 1 is an upper bound of h.

always positive because $x > y^n$

$hn\,s^{n-1} < \left(\frac{x-y^n}{n(y+1)^{n-1}}\right)n(y+1)^{n-1}$

$hn\,s^{n-1} < x - y^n$

by transitivity:

$s^n - y^n < x - y^n$

$s^n < x$ (OF +) (Inv +)

i. this implies that a number bigger than y, namely $s^n = (y+h)^n$ is not an upper band of A. Since we set y as the suprema of A, we cannot have $x > y^n$

j. Since $y^n > x$ and $y^n < x$ cannot be the case, and we are in a complete ordered field where $u \leq t$ or $u \geq t$ or $u = t$ must be the case, $y^n = x$.